8.5 Applications of Matrices and Determinants

Objectives: SWBAT use Cramer's Rule to solve a system of equations, use determinants to find the area of triangles, and use matrices to encode and decode messages.

### Bellwork:
Find the inverse *by hand*.

\[
\begin{bmatrix}
4 & -3 \\
-7 & 6
\end{bmatrix}
\]

\[
\begin{vmatrix}
4 & -3 \\
-7 & 6
\end{vmatrix} = 21 - 21 = 0
\]

\[
\frac{1}{3} \begin{bmatrix}
6 & 3 \\
7 & 4
\end{bmatrix} = \begin{bmatrix}
2 & \frac{1}{3} \\
\frac{7}{3} & \frac{1}{3}
\end{bmatrix}
\]

*on calculator*

\[
\begin{bmatrix}
3 & 0 & 4 \\
-5 & -2 & 6 \\
-1 & 5 & -2
\end{bmatrix}
\]

\[
\begin{vmatrix}
3 & 0 & 4 \\
-5 & -2 & 6 \\
-1 & 5 & -2
\end{vmatrix} = 24 - 4 = 20
\]

\[
\begin{bmatrix}
3 & 0 \\
-5 & -2 \\
-1 & 5
\end{bmatrix}^{-1} = \frac{1}{20} \begin{bmatrix}
2 & 0 \\
5 & 4 \\
1 & 5
\end{bmatrix}
\]

### Solving Systems of Linear Equations Using Inverses

If the coefficient matrix A of a square system (a system with the same number of equations and unknowns) is invertible, there exists a unique solution.

If A is an invertible matrix, represented as \(AX = B\), then the linear system has a unique solution given by \(X = A^{-1}B\).
Example 5: Solve the system of equations.

\[
\begin{align*}
3x + 4y &= -2 \\
5x + 3y &= 4
\end{align*}
\]

Solving a system using an inverse:
Given a system of linear equations: \(AX = B\)

Find the unique solution: \(X = A^{-1}B\)

You have $10,000 to invest. You want to invest the money in a stock mutual fund, a bond mutual fund, and a money market fund. The expected annual returns for these funds are given in the table.

<table>
<thead>
<tr>
<th>Investment</th>
<th>Expected Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock mutual fund</td>
<td>10%</td>
</tr>
<tr>
<td>Bond mutual fund</td>
<td>7%</td>
</tr>
<tr>
<td>Money Market fund</td>
<td>5%</td>
</tr>
</tbody>
</table>

You want your investment to obtain an overall annual return of 8%. A financial planner recommends that you invest the same amount in stocks as in bonds and money markets combined. How much should you invest in each fund?
Cramer’s Rule: uses determinants to solve systems of linear equations.

Given the system:
\[
\begin{align*}
ax + by &= e \\
(cx + dy) &= f
\end{align*}
\]

\[
\begin{vmatrix}
a & e \\
b & f
\end{vmatrix}
\]

\[
\begin{vmatrix}
c & f \\
d & e
\end{vmatrix}
\]

Example 1:
Use Cramer’s Rule to solve (if possible) the system of equations.

\[
\begin{align*}
-7x + 11y &= -1 \\
3x - 9y &= 9
\end{align*}
\]

\[
\begin{vmatrix}
-7 & 11 \\
3 & -9
\end{vmatrix} = (63x - 99) = 30
\]

\[
\begin{vmatrix}
-7 & 11 \\
3 & -9
\end{vmatrix} = \frac{-3}{2}
\]

\[
\begin{vmatrix}
-7 & 11 \\
3 & -9
\end{vmatrix} = \frac{-3}{2}
\]

Example 1:
Use Cramer’s Rule to solve (if possible) the system of equations.

\[
\begin{align*}
4x - 2y &= 10 \\
3x - 5y &= 11
\end{align*}
\]

\[
\begin{vmatrix}
4 & -2 \\
3 & -5
\end{vmatrix} = 4(-5) - (-2) = 14
\]

\[
\begin{vmatrix}
4 & -2 \\
3 & -5
\end{vmatrix} = 2
\]

\[
\begin{vmatrix}
4 & -2 \\
3 & -5
\end{vmatrix} = -1
\]

\[
\begin{vmatrix}
4 & -2 \\
3 & -5
\end{vmatrix} = (2)\cdot(-1)
\]

Example 1:
Use Cramer’s Rule to solve (if possible) the system of equations.

\[
\begin{align*}
-0.4x + 0.8y &= 1.6 \\
0.2x + 0.3y &= 2.2
\end{align*}
\]

\[
\begin{vmatrix}
-0.4 & 0.8 \\
0.2 & 0.3
\end{vmatrix} = (-0.12) - (0.08) = -0.2
\]

\[
\begin{vmatrix}
-0.4 & 0.8 \\
0.2 & 0.3
\end{vmatrix} = \frac{32}{7}
\]

\[
\begin{vmatrix}
-0.4 & 0.8 \\
0.2 & 0.3
\end{vmatrix} = \frac{-0.28}{7}
\]

\[
\begin{vmatrix}
0.2 & 0.3 \\
0.2 & 0.3
\end{vmatrix} = \frac{0.03}{7}
\]

\[
\begin{vmatrix}
0.2 & 0.3 \\
0.2 & 0.3
\end{vmatrix} = \frac{0.02}{7}
\]
Example 2: Use Cramer’s Rule to solve (if possible) to solve the 3x3 system of equations:

\[
\begin{align*}
-x + 2y - 3z &= 1 \\
2x + z &= 0 \\
3x - 4y + 4z &= 2
\end{align*}
\]

Area of a Triangle:
The area of a triangle with vertices \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) is:

\[
\text{Area} = \frac{1}{2} \left| x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 \right|
\]

Where the symbol \(\pm\) indicates that the appropriate sign should be chosen to yield a positive area.

Example 3: Use a determinant and the given vertices of a triangle to find the area of the triangle.

\[
\begin{align*}
\text{Area} &= \frac{1}{2} \left| \begin{array}{cc}
1 & 5 \\
0 & 4 \\
3 & 1
\end{array} \right| \\
&= \frac{1}{2} \left( 1 \cdot 4 - 5 \cdot 3 \right) \\
&= \frac{1}{2} \left( 4 - 15 \right) \\
&= \frac{1}{2} \left( -11 \right) \\
&= -\frac{11}{2}
\end{align*}
\]

Cryptography:

\[
\begin{align*}
0 &= _{A} \\
1 &= A \\
2 &= B \\
3 &= C \\
4 &= D \\
5 &= E \\
6 &= F \\
7 &= G \\
8 &= H \\
9 &= I \\
10 &= J \\
11 &= K \\
12 &= L \\
13 &= M \\
14 &= N \\
15 &= O \\
16 &= P \\
17 &= Q \\
18 &= R \\
19 &= S \\
20 &= T \\
21 &= U \\
22 &= V \\
23 &= W \\
24 &= X \\
25 &= Y \\
26 &= Z
\end{align*}
\]
**Encoding a Message:**

1.) Break up the message into groups of 3 (including blank spaces, but ignoring punctuation).
2.) Multiply each group of 3 by the given matrix A (hint: use your calculator)
3.) Gives you a new 1 x 3 matrix which is your encoded message and you write it as a string of numbers without the matrix notation.

**Example 4:**
Write a cryptogram for the message using Matrix A.

#5 LANDING SUCCESSFUL 
A = \[
\begin{bmatrix}
1 & 2 & 2 \\
3 & 7 & 9 \\
-1 & -4 & -7
\end{bmatrix}
\]

#6 HAPPY BIRTHDAY

**Decoding a Message:**

1.) Find the inverse of Matrix A (hint: use your calculator)
2.) Then break up the encoded message into groups of 3.
3.) Multiply the 1 x 3 matrices by the inverse of Matrix A which gives you your decoded message.

4.) Write as a string of numbers and apply the chart from CRYPTOGRAPHY slide!

**Example 5:**
Use A⁻¹ to decode the cryptogram.

A = \[
\begin{bmatrix}
1 & 2 \\
3 & 5
\end{bmatrix}
\]

11, 21, 64, 112, 25, 50, 29, 53, 23, 46, 40, 75, 55, 92
Example 5:
Use $A^{-1}$ to decode the cryptogram.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

9, -1, -9, 38, -19, -19, 28, -9, -19, -80, 25, 41, -64, 21, 31, 9, -5, -4

8.5 assignment
p. 606 # 51, 54, 65
p. 625 # 8, 9, 13, 15, 22, 26, 33, 37

homework quiz Monday!